Indian Statistical Institute, Bangalore
B.Math (Hons.)II Year, First Semester
Semestral Examination
Algebra - III

Time: 3 hours Date: Nov 26 2010 Instructor: N.S.N.Sastry

Maximum Marks 100

Answer all questions. Your answers should be complete and to the point.

1. Define elementary symmetric polynomials in n variables over a field k. Show that they generate the ring of symmetric functions in n variables over k. [10]

2. Let $T_4(k)$ be the ring of upper triangular matrices of order 4 with entries from a field k. Determine the submodules of the $T_4(k)-$ module k^4 (with natural module structure). [10]

3. With justification, give an example for each of the following:

   (a) An indecomposable, but not irreducible, $\mathbb{Z}$- module.

   (b) An example of an integral domain which is not a unique factorization domain.

   (c) A ring in which every ideal is finitely generated but not necessarily generated by a single element.

   (d) A ring which is neither Noetherian nor Artinian. [5+5+8+7 = 25]

4. (a) For a ring R and a natural number n, define a free R- module of rank n.

   (b) Using the definition of a free R- module, show that:

   (i) $Q/\mathbb{Z}$ is not a free $\mathbb{Z}$ - module.

   (ii) any R- module generated by n elements is a homomorphic image of a free R- module of rank n. [5+8+7 = 20]

5. Define a Noetherian ring. If R is a Noetherian ring and S is a multiplicative subset of R, then show that the localization of R at S is also Noetherian.

   [3+7]

6. (a) Show that, for $a,\ b \in\ \mathbb{Z}, a \neq 0$, there is a unique ring automorphism f of $\mathbb{Z}[X]$ such that $f(X) = aX + b$.

   (b) Show that any ring automorphism θ of $\mathbb{Z}[X]$ is defined by $\theta(X) = pX + q$ for some $p,\ q\ \in\ \mathbb{Z}, p \neq 0$. [7+8]

7. (a) Construct a field of order 27.

   (b) Does there exist a field of order 2010. Justify your answer. [6+4]

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